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7/25/68

AN ANALYTICAL STUDY OF THE DYNAMIC RESPONSE OF THE INFINITE BERNOULLI-EULER BEAM WITH DAMPING AND AN ELASTIC FOUNDATION

A THESIS

Presented to

The Faculty of the Graduate Division

by

Wolfram Stadler

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Engineering Mechanics

Georgia Institute of Technology

August, 1968

AN ANALYTICAL STUDY OF THE DYNAMIC RESPONSE OF

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Approved:

Date approved by Chairman: 10/2/'68

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Holly Fenning

ACKNOWLEDGMENTS

I am grateful to my advisor, Dr. R. W. Shreeves, for initially posing this problem, and for numerous discussions in connection therewith.

Furthermore, I appreciate the patience expressed by Dr. W. W. King, in listening to and criticizing the many discourses presented to him concerning this problem.

My thanks also to Dr. J. M. Osborn for criticizing and refining the mathematics involved in this study.

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SUMMARY

A closed form analytical solution is obtained for the dynamic response of an infinite Bernoulli-Euler beam subjected to an arbitrary load, with arbitrary initial conditions. The simultaneous effects of damping, an elastic foundation and a constant axial load are considered. The dependence of the solution on the parameters describing these effects is quite obvious from the form of the solution; thus, it is possible to define critical, subcritical and supercritical damping similar to the definitions given for a damped spring-mass system. Solutions of infinite beam problems from the literature are shown to be special cases of the general solution given here. In particular, it is shown that the steady state solutions dealt with in several papers may be obtained by a formal passage to the limit as time tends to infinity in the transient solution. Solutions to new problems are presented as examples by introducing particular forcing functions into the general solution.

INTRODUCTION

The problem of the free and forced vibration of an infinite Bernoulli-Euler beam is solved by the use of transform methods. The effects of damping, an elastic foundation and a constant axial load are included. In essence then, this amounts to the construction of the general solution of the partial differential equation

$$EI \frac{\partial^4 w}{\partial x^4} \pm S \frac{\partial^2 w}{\partial x^2} + kw + d \frac{\partial w}{\partial t} + \rho \frac{\partial^2 w}{\partial t^2} = q$$

for

$$-\infty < x < \infty \quad \text{and} \quad t > 0,$$

subject to suitable initial and boundary conditions. Here, $w(x,t)$ represents the deflection, $q(x,t)$ the lineal loading, EI is the flexural rigidity and S is a constant axial force (+ for compression, - for tension). The constants k, d, and ρ represent the foundation constant, the damping coefficient, and the lineal mass, respectively.

The analysis of infinite beams dates back to Fourier* [1818], who first solved the problem of free vibration of a simple** infinite beam, subject to particular initial conditions. An account of the available

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*Todhunter, I. and Pearson, K.: A History of the Theory of Elasticity and of the Strength of Materials. Vol. I, Dover, New York, 1960, p. 112

**Throughout the remainder of this thesis "simple" shall imply $S = k = d = 0$.

literature concerning the steady state response of infinite beams and solutions of transient problems is given in the next section.

In this thesis the transient solution to the subject problem is obtained. Care is taken to state all assumptions in the physical and the mathematical formulation of the problem. Detailed justifications of the mathematical steps employed in constructing the general solution are given in the appendices. Problems in the literature are shown to be special cases of this general solution. The steady state solutions in the literature were of particular interest. Usually, the steady-state solution is defined simply as the condition which prevails for large t, or that the solution becomes time-independent. These definitions are vague, and the difficulty here is probably best expressed by J. J. Stoker,* who writes: "...that the difficulty arises because the steady-state problem is an unnatural problem in mechanics and that, in principle at least, one should rather formulate and solve an initial-value problem and then find the solution of the time-independent problem by making a passage to the limit in allowing the time to tend to infinity." It is thus of interest to show that the steady-state solutions in the literature can be obtained as the result of passages to the limit as t tends to infinity in the transient solution. The term "transient solution" throughout this thesis will refer simply to the solution of the initial value problem with appropriate forcing function.

In addition to the establishment of correspondence with existing solutions, several new problems are solved as examples of the application

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*Stoker, J. J.: On Radiation Conditions. Communications on Pure and Applied Mathematics, Vol. IX, 1956, pp. 577-595.

of the general solution.

With the existence of this general solution several new avenues for the investigation of infinite beams are opened, among which are the treatment of a mass moving on the damped, infinite beam on an elastic foundation, and the question of resonance effects due to various time-dependent loads.

CHAPTER I

HISTORICAL INTRODUCTION

It seems certain that the first man who hit a wooden rod with another for musical or other purposes must have noted that there were points of support which made the resulting sound clearer. However, the first written account of attempts to make calculations concerning the vibration of rods was made by Christaan Huygens* [1688], who considered a bar supported so as to minimize the "danger of rupture." He wrote that a bar supported in such a manner gave the clearest sourd when struck, and that, in effect, these points remained at rest (i.e. he endeavored to find the nodes). These were experimental calculations - the physical and mathematical models describing the phenomenon were still lacking. Not until James and Daniel Bernoulli provided the curvature-bending relationships and Leibnitz the integral calculus, were the necessary tools available. It was not long before Daniel Bernoulli and Leonhard Euler** [1735] then used these tools to obtain for the first time, independently, the equation of motion for an elastic lamina. Since Euler's derivation is more readily available, it shall be given here:

As early as 1704 James Bernoulli resolved the forces acting on a plane flexible line into rectangular components. Until then it had been

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*Truesdell, C.: The Rational Mechanics of Flexible or Elastic Bodies 1638-1788. Introduction to Leonhardi Euleri Opera Omnia, vol. X et XI, Seriei Secundae, Orell Füssli, Turici 1960, p. 49.

**Ibid., p. 168.

usual to use tangential and normal components. Euler* [1728] wrote the moment equation of an elastica as

$$-P_y x + P_x y - \int_o^x Y\,dx + \int_o^y X\,dy = -\frac{B}{r} \tag{1.1}$$

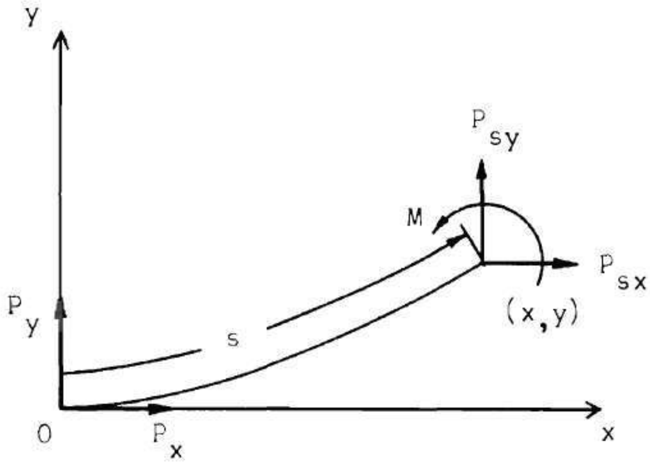

Figure 1. Equilibrium of the Elastica.

by taking moments about an arbitrary point (x,y) on the elastica. In the equation, P_x and P_y denote the forces in the subscripted directions, r is the radius of curvature at (x,y), B is "the absolute elasticity" (in Euler's words) and

$$Y = \int_o^s F_y ds \ , \quad X = \int_o^s F_x ds \ . \tag{1.2}$$

F_x and F_y are distributed forces per unit length. From the observations of a simple pendulum Euler deduced that in any configuration**

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*<u>Ibid.</u>, p. 148.

**<u>Ibid.</u>, p. 163.

$$\frac{\text{accelerating force}}{\text{weight}} = \frac{\text{displacement}}{\text{length of pendulum}} . \qquad (1.3)$$

In 1735, he set $P_x = P_y = F_x = 0$ in equation (1.1) and used (1.3) to write the force due to the gravitational acceleration g as

$$F_y = \frac{\rho g y}{\alpha} , \qquad (1.4)$$

where ρ is the lineal mass density and α the length of an isochronous pendulum. The substitution of (1.4) in (1.1) leads to

$$\frac{B}{r} = B \frac{d^2 y}{dx^2} = \frac{g}{\alpha} \int_o^x dx \int_o^x \rho y dx , \qquad (1.5)$$

where small deformations have now been assumed. The equation corresponds to an elastic lamina fixed at one end and oscillating in a vertical plane. Two differentiations with respect to x result in

$$K^4 \frac{d^4 y}{dx^4} = y, \quad (K^4 = \frac{\alpha B}{\rho g}) . \qquad (1.6)$$

It is interesting at this point to compare this method of derivation with the methods used today. The equivalent partial differential equation is

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{EI} \frac{\partial^2 y}{\partial t^2} = 0,$$

which is solved by assuming

$$y(x,t) = e^{i\omega t} Y(x),$$

where

$$\frac{d^4Y}{dx^4} - \frac{\omega^2\rho}{EI} Y = 0 .$$

A comparison of the coefficient in this equation with K gives

$$\omega^2 = \frac{g}{a} ,$$

and it is now clear what was meant by an equivalent isochronous pendulum.

Euler continued his derivation by considering a bar of length ℓ and solving (1.6) subject to

$$y = A, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0 \quad \text{at} \quad x = 0 , \tag{1.7}$$

with resultant frequencies

$$\nu = \frac{\xi^2}{2\pi\ell^2}\sqrt{\frac{B}{\rho}} . \tag{1.8}$$

The ξ's are the solutions of the transcendental equation

$$1 + \cos\xi \cosh\xi = 0 , \tag{1.9}$$

resulting from the condition

$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = \ell .$$

Euler calculated only the fundamental frequency and made no comment concerning the fact that there were infinitely many. It is quite probable that he considered this obvious, and unworthy of mention.

It is noteworthy that until this time only total derivatives appeared in the literature. Euler had obtained a partial differential equation as early as 1734 in his investigations of some geometrical problems, but the first partial differential equation, subjected to intensive study was the wave equation derived and solved by D'Alembert* [1746]. The theory of partial differential equations was properly founded only when Euler supplied the calculus of partial derivatives in a series of papers on mechanical subjects during the years 1748-1766. Euler** [1750-57] was the first to write the equation for the small transverse oscillation of an elastic rod as

$$\frac{1}{c^4}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0, \quad \left(c^4 = \frac{EI}{\rho}\right) \tag{1.10}$$

a form, which is still in use. The effect of an axial, tensile force S was included, again by Euler***[1772], i.e.

$$-S\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial^2 y}{\partial t^2} = -B\frac{\partial^4 y}{\partial x^4}, \tag{1.11}$$

where he noted that the expression reduces to the equation for the string when B = 0.

The first treatment of the problem of an infinite rod probably was that given by Fourier****[1818], who referred to Euler's memoir of 1779 to solve

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*Ibid., p. 237.

**Ibid., p. 254.

***Ibid., p. 327

****Todhunter and Pearson, Op. cit., Vol. I, p. 112.

$$\frac{\partial^2 z}{\partial t^2} + a^2 \frac{\partial^4 z}{\partial x^4} = 0, \qquad (a^2 = \frac{EI}{\rho}) \tag{1.12}$$

subject to the initial conditions

$$z = \Phi(x), \quad \frac{\partial z}{\partial t} = 0 \ , \tag{1.13}$$

with the result

$$z = \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{\infty} \sin\left(\frac{\pi}{4} + \frac{(\xi - x)^2}{4at}\right)\Phi(\xi)d\xi \ . \tag{1.14}$$

Boussinesq* [1882] extended this solution to initial conditions of the form

$$w = F(x) \quad \text{and} \quad \frac{\partial w}{\partial t} = a\ F_1''(x) \tag{1.15}$$

to obtain

$$w = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [F(x + 2\alpha\sqrt{at})(\sin \alpha^2 + \cos \alpha^2)$$

$$+ F_1(x + 2\alpha\sqrt{at})(\sin \alpha^2 - \cos \alpha^2)]d\alpha \ . \tag{1.16}$$

The primes indicate differentiation with respect to x and w is equivalent to z in equation (1.12). For the moment this concluded the efforts expended on infinite beams. During the next 60 years nothing appeared in the literature on infinite beams; however, the response of finite beams to moving and nonmoving loads was studied extensively. For completeness some of the more pertinent works are cited here. Bühler [1909]

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*Ibid., vol. II, p. 282.

gave a summary of the work done concerning the vibration of finite beams up until 1909. Jeffcott [1929] and Inglis [1934] obtained the effects of a point mass moving with a constant velocity by using methods of successive approximations and harmonic analysis, respectively.

Finally, Prager [1933] illustrated the application of symbolic methods by obtaining the moment distribution corresponding to a concentrated moment applied at $x = 0$, remaining constant thereafter, and due to an impulsive force applied at $t = 0$, $x = 0$. Apparently the first one to deal with an infinite beam on an elastic foundation was Ludwig [1938], who obtained the steady-state solution corresponding to

$$EI \frac{\partial^4 y}{\partial x^4} + cy + \rho \frac{\partial^2 y}{\partial t^2} = F\,\delta(x - vt) \tag{1.17}$$

by assuming that "the deflection profile will move with the same velocity as the load and will be independent of time," i.e. the deflection becomes

$$y = w(\eta), \quad \eta = x - vt\,, \tag{1.18}$$

where w now is the solution of

$$EI \frac{d^4 w}{d\eta^4} + cw + \rho v^2 \frac{d^2 w}{d\eta^2} = F\,\delta(\eta)\,, \tag{1.19}$$

obtained from equation (1.17) by substituting the assumption (1.18). In these equations EI is the flexural rigidity, c the foundation constant, ρ and F are the lineal mass and force, respectively, v is the velocity of the load, and δ is the Dirac Delta "function." To construct the solution for the infinite beam, Ludwig added Timoshenko's [1926] solutions of the forced (moving load) and free vibrations of a simply-supported beam of length L, and subsequently let L tend to infinity. It is of interest that this is the only paper in which the solution for an infinite beam is obtained from that of a finite beam by letting L tend to infinity. His result was

$$w_{\text{behind load}} = \frac{F}{4\sqrt{EI\,c}}\, e^{\beta(x-vt)}\left[-\frac{\cos\gamma(x-vt)}{\beta} + \frac{\sin\gamma(x-vt)}{\gamma}\right],$$

$$w_{\text{ahead of load}} = \frac{F}{4\sqrt{EI\,c}}\, e^{-\beta(x-vt)}\left[-\frac{\cos\gamma(x-vt)}{\gamma} + \frac{\sin\gamma(x-vt)}{\gamma}\right], \qquad (1.20)$$

valid for $v < v_{cr}$, with

$$\beta = \sqrt{\sqrt{\frac{c}{4EI}} - \frac{\rho v^2}{4EI}} \quad \text{and} \quad \gamma = \sqrt{\sqrt{\frac{c}{4EI}} + \frac{\rho v^2}{4EI}}\ . \qquad (1.21)$$

Ludwig defined

$$v_{cr} = \sqrt[4]{\frac{4cEI}{\rho^2}}\ , \qquad (1.22)$$

which he termed the propagation velocity. He also discussed the additional cases $v = v_{cr}$ and $v > v_{cr}$ and gave solutions. Unfortunately, the results he obtained for these cases were incorrect, as was pointed out by Dörr [1943], who, in his doctoral dissertation, discussed both the steady-state solution and the transient solution corresponding to equation (1.17) with F = 1 and zero initial conditions. Dörr wrote his solution in the form

$y = y_\infty + y_h$. ($y_\infty = \lim_{t \to \infty} y$, y_h - solution of the homogeneous equation)

that is he defined the transient solution as that part of the total solution which tends to zero as t tends to infinity. This is not the same definition which will be used throughout this thesis, and which was given in the introduction. Since both Ludwig's and Dörr's papers are often cited but seldom discussed, and in view of the incorrectness of part of Ludwig's solution, it may be of interest to review Dörr's paper in more detail.

Dörr began with a discussion of the possible wave motions by assuming, as is usual

$$y = e^{i(kx - \omega t)} \tag{1.23}$$

with a phase velocity defined by

$$\varphi(k) = \frac{\omega(k)}{k} = \sqrt{\frac{EI}{\rho} k^2 + \frac{c}{\rho k^2}} \ , \tag{1.24}$$

and a corresponding minimum phase velocity

$$\varphi_{min} = v_{cr} = \sqrt[4]{\frac{4cEI}{\rho^2}} \ .$$

He arrived at the conclusion that the character of the solution had to be fundamentally different for the two cases $v < v_{cr}$ and $v > v_{cr}$. To solve the problem he represented y in terms of an integral which was related to the Fourier integral. The major part of his paper was concerned with the steady-state solution ($t \to \infty$). He obtained the following results (represented schematically in Figure 2):

(a) For $v < v_{cr}$ the solution is the same as that obtained by Ludwig, that is the steady-state solution depends on t only in the combination x - vt. This result Dörr accomplished by using asymptotic expressions (valid for large t) for integrals which occurred in his representation of y.

(b) For $v > v_{cr}$ he concluded that a steady-state solution in the sense of (a) did not exist. Instead, he discovered that there exists a wave train with a constant amplitude A_1 and constant wavelength b_1 in front of the load, and a similar one behind the load with amplitude A_2 and wavelength b_2. The length of the total wave train is a function of time, i.e. the wave head (constants A_1, b_1) moves with the group-velocity of the wave packet in front of the load,

$$V_1 = v\left(1 + \sqrt{1 - \frac{4cEI}{\rho^2 v^4}}\right), \tag{1.25}$$

and the wave tail (constants A_2, b_2) moves with the group velocity of the wave packet behind the load,

$$V_2 = v\left(1 - \sqrt{1 - \frac{4cEI}{\rho^2 v^4}}\right). \tag{1.26}$$

Both amplitudes, A_1 and A_2 decay rapidly with the distance from the load, when this distance extends beyond the wave head and wave tail. This decay is indicated by the dashed lines in Figure 2b.

(c) In case (a) he obtained the transient solution in terms of an integral of the steady-state solution (y_∞) by assuming

$$y = y_\infty + y_h$$

with zero initial conditions. For the transient solution in case (b) he returned to an intermediate integral and obtained an asymptotic series representation valid for all v, including $v = v_{cr}$, and for moderately large t. He stated, with regard to this solution: "The coefficients of this series have such a complicated generating function, that general conclusions therefrom are doubtful."

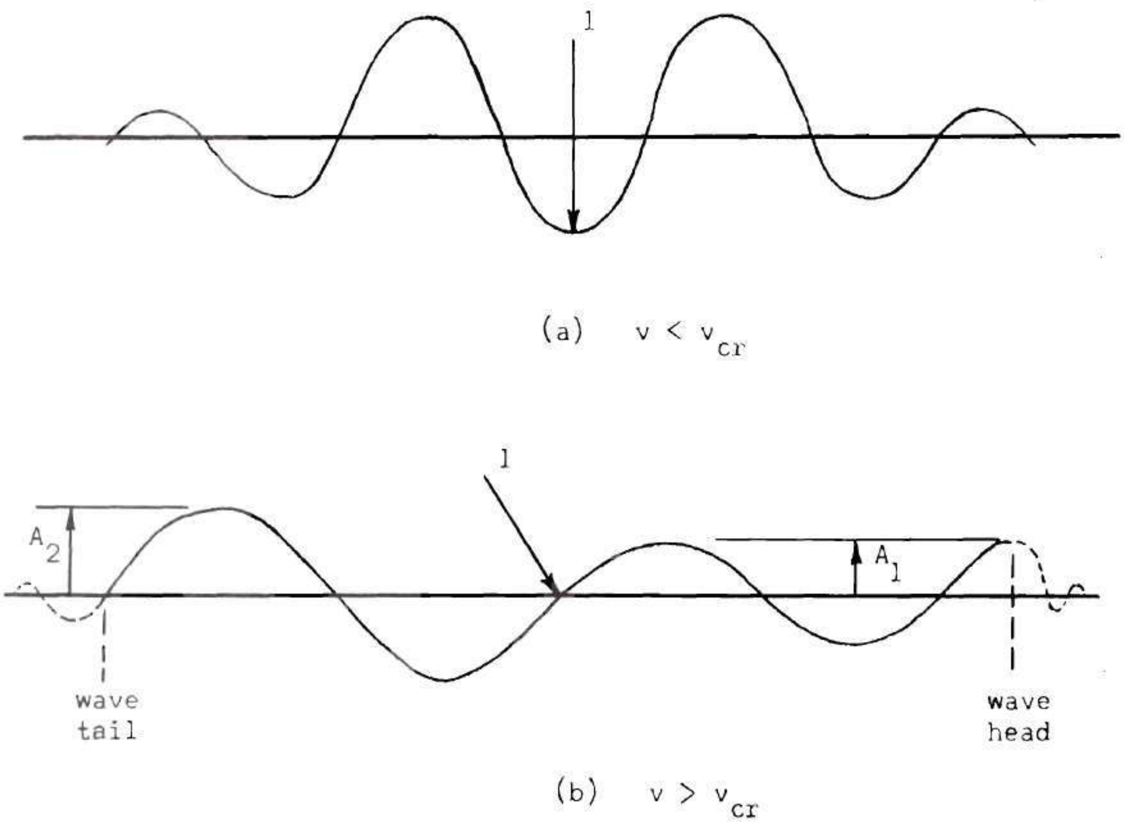


Figure 2. Schematic Representations of Dörr's Steady State Solutions (as they appear in Dörr's paper).

Dörr concluded his discourse with energy considerations involving the steady-state solutions, and with comparisons to the solutions of static problems.

It is apparent from the cited examples that damping was always assumed to be present but was not included explicitly in the analysis. J. T. Kenney [1954] finally included the effects of damping and obtained the steady-state solution of equation (1.17) in the form (1.19). This solution involved a parameter η, which was the root of a sixth degree polynomial, whose coefficients involved the material parameters. P. M. Mathews [1958] obtained the steady-state solution due to a time dependent moving force, namely, he solved

$$EI \frac{\partial^4 y}{\partial x^4} + ky + \rho \frac{\partial^2 y}{\partial t^2} = F_o(\cos \omega t)\delta(x-vt) \tag{1.27}$$

by assuming the steady-state solution had the form

$$y(r,t) = y_1(r)\cos \omega t + y_2(r) \sin \omega t, \tag{1.28}$$

where

$$r = x - vt .$$

Again the solution depended on a parameter, which, in this case was one of the roots of a fourth degree polynomial, with the result that the dependence of the solution on the material parameters was once more obscured. He initially included damping, but then decided to solve the "much simpler" case with the damping set equal to zero. An additional result here was that there were combinations of ω and v for which the beam was observed to exhibit resonance phenomena.

The problem of a mass moving on a simple infinite beam was solved by C. Tseng [1962]. In the process he obtained the Green's function corresponding to

$$EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} = F(t)\delta(x-\xi) \tag{1.29}$$

in the form

$$G(x,y;\ \xi,\eta) = (x-\xi)\left\{\frac{\sqrt{2\pi}}{\nu}\left[\cos\frac{\nu^2}{4} + \sin\frac{\nu^2}{4}\right] + \pi\left[\mathcal{S}_T\left(\frac{\nu^2}{4}\right) - \mathcal{C}_T\left(\frac{\nu^2}{4}\right)\right]\right\}, \tag{1.30}$$

where

$$\nu = \frac{x-\xi}{(y-\eta)^{1/2}} .$$

W. Nowacki [1963] considered an infinite rod on an elastic foundation subjected to a loading

$$q(x,t) = P_o\ \delta(x)\ \delta(t) . \tag{1.31}$$

However, he solved the problem only for the case $x = 0$, for which he calculated

$$y(0,t) = \frac{P_o\Gamma(\frac{3}{4})}{2\sqrt{\pi}\sqrt[4]{4EI}\ (2k)^{3/4}}\left(\sqrt{\frac{k}{\rho}}\,t\right)^{\frac{1}{4}} J_{1/4}\left(\sqrt{\frac{k}{\rho}}\,t\right). \tag{1.32}$$

C. C. Fu [1967] obtained a closed-form solution for a step velocity "applied suddenly" to a simple infinite beam. He constructed his solution by assuming a suddenly imparted displacement in the form of a power series in time.

At this point, the only thing which had not been done to a simple infinite beam was the consideration of random loads. J. K. Knowles [1968] abolished this deficiency by solving the problem of a constant load whose position on the beam at a time t was $x = X(t)$, where $X(t)$ was a random process.

CHAPTER II

PROBLEM FORMULATION

2.1 Physical Formulation

The usual assumptions for elementary beam theory are made. In addition there are assumptions concerning the type of foundation and damping present in the system. For completeness, all of them will be listed. They are:

(i) The medium is homogeneous and isotropic.

(ii) The medium is Hookean.

(iii) The beam experiences only small deflections.

(iv) The beam has a neutral axis.

(v) Plane sections perpendicular to the neutral axis before deformation remain plane and perpendicular to the neutral axis after deformation.

(vi) The cross-section of the beam is constant.

(vii) Deflection of each particle of the cross-section is the same as that of the center of mass of the cross-section.

(viii) The resistance of the foundation is a linear function of the deflection (Winkler's assumption).

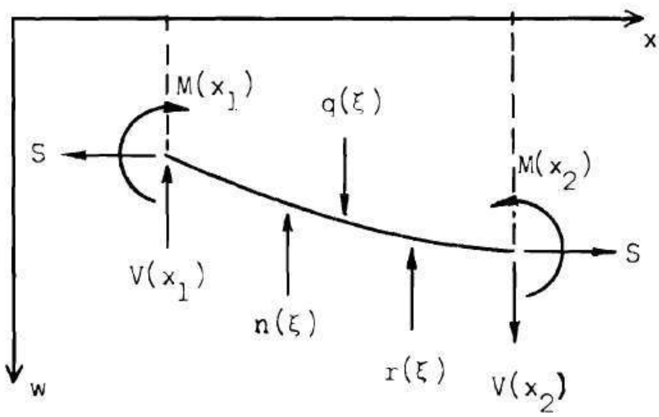
(ix) The damping inherent in the foundation is linearly viscous.

2.2 Mathematical Formulation

In the derivation of the applicable partial differential equation the sign convention depicted in Figure 3 is used. The equation of a beam subjected to a constant tensile axial load S and a distributed load $p(x,t)$ is

$$EI \frac{\partial^4 w}{\partial x^4} - S \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} = p(x,t) \ , \tag{2.2.1}$$

where EI is the flexural rigidity, $w(x,t)$ is the deflection, and ρ and $p(x,t)$ are the lineal mass density and loading, respectively.



$$x_1 \leq \xi \leq x_2$$

Figure 3. Sign Convention.

For the problem under consideration

$$p(x,t) = q(x,t) - r(x,t) - n(x,t) \ , \tag{2.2.2}$$

where $q(x,t)$ is an arbitrary load, $r(x,t)$ is the resistance of the foundation, and $n(x,t)$ is the damping inherent in the foundation; all of them are taken to be per unit length of the beam. When the assumptions (viii) and (ix) of the preceding section are included, the final form of equation (2.2.1) is

$$EI \frac{\partial^4 w}{\partial x^4} - S \frac{\partial^2 w}{\partial x^2} + kw + d \frac{\partial w}{\partial t} + \rho \frac{\partial^2 w}{\partial t^2} = q \, , \tag{2.2.3}$$

where k is the foundation constant and d is the damping coefficient. Equation (2.2.3) is to be solved for

$$- \infty < x < \infty, \quad t > 0$$

subject to:

(i) The initial conditions

$$\lim_{t \to 0} w(x,t) = f(x), \quad \lim_{t \to 0} \frac{\partial w(x,t)}{\partial t} = g(x) \, . \tag{2.2.4}$$

(ii) The order requirements

$$\lim_{|x| \to \infty} \frac{\partial^\nu w(x,t)}{\partial x^\nu} = 0, \quad \text{for} \quad \nu = 0,1,2,3, \tag{2.2.5}$$

where it is understood that $\nu = 0$ corresponds to w itself.

This concludes the mathematical formulation of the problem.

CHAPTER III

SOLUTION OF THE PROBLEM

3.1 Construction of the Solution

Transform methods are employed to establish the solution. A Fourier transformation of equation (2.2.3) with respect to x results in an ordinary differential equation in t. A subsequent Laplace transformation with respect to t results in an algebraic equation for the transformed deflection $\bar{W}(\alpha,p)$ [A1].* The inversion of both transforms then yields the final result. In order to proceed with the solution in the indicated manner, the following additional assumptions must be made:

(i) The Laplace transforms with respect to t and the Fourier transforms with respect to x of $w(x,t)$ and $q(x,t)$ exist.

(ii) The functions f and g have Fourier transforms.

(iii) The Fourier transformation of w with respect to x is interchangeable with the differentiation with respect to t, i.e.,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\mathcal{F}\{w\}\right] = \frac{dW}{dt} ,$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[\mathcal{F}\{w\}\right] = \frac{d^2 W}{dt^2} .$$

(iv) The Fourier transformation of w with respect to x is interchangeable with the limit operation on t, i.e.

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*The numbers prefixed by A and enclosed in brackets denote the correspondingly numbered section in the appendix.

$$\mathcal{F}\left\{\lim_{t\to 0} w(x,t)\right\} = \lim_{t\to 0} \mathcal{F}\left\{w(x,t)\right\} = \lim_{t\to 0} W(\alpha,t) = W(\alpha,0^+) = F(\alpha),$$

$$\mathcal{F}\left\{\lim_{t\to 0} \frac{\partial w(x,t)}{\partial t}\right\} = \lim_{t\to 0} \mathcal{F}\left\{\frac{\partial w(x,t)}{\partial t}\right\} = \lim_{t\to 0} \frac{dW(\alpha,t)}{dt} = \frac{dW(\alpha,0^+)}{dt} = G(\alpha).$$

Normally the solution w is sought as a member of a class of functions satisfying certain continuity and smoothness conditions. This class clearly depends on the smoothness and continuity of f, g, and q. However, once f, g and q are given the class may be stipulated. If then, it turns out, that w is a member of this class, the assumptions (i) - (iv) above may be dropped, since, as long as it can be shown that w is a solution satisfying the stipulated conditions, the manner of arriving at the solution is irrelevant.

In view of the assumptions (i) - (iv) and the order requirements (2.2.5) the Fourier transform of equation (2.2.3) with respect to x is

$$EI\,\alpha^4 W + S\alpha^2 W + kW + d\,\frac{dW}{dt} + \rho\,\frac{d^2W}{dt^2} = Q\,, \tag{3.1.1}$$

now subject to

$$\lim_{t\to 0} W(\alpha,t) = F(\alpha), \quad \text{and} \quad \lim_{t\to 0} \frac{dW(\alpha,t)}{dt} = G(\alpha). \tag{3.1.2}$$

The subsequent Laplace transform with respect to t is

$$EI\alpha^4\,\bar{W} + S\alpha^2\,\bar{W} + k\,\bar{W} + dp\,\bar{W} + \rho p^2\,\bar{W} = (d + \rho p)F + \rho G + \bar{Q}\,. \tag{3.1.3}$$

This equation is solved for $\bar{W}$, with the result

$$\bar{W}(\alpha,p) = \frac{(d + \rho p)F + \rho G + \bar{Q}}{EI\alpha^4 + S\alpha^2 + k + dp + \rho p^2}\,. \tag{3.1.4}$$

A form of $\bar{W}$ more suitable for the inversion of the transforms is

$$\bar{W}(\alpha,p) = \frac{\zeta F + G}{(p+\zeta)^2 + a^2(\alpha^2+b)^2 + \lambda^2} + \frac{(p+\zeta)F}{(p+\zeta)^2 + a^2(\alpha^2+b)^2 + \lambda^2}$$

$$+ \frac{1/\rho \ \bar{Q}}{(p+\zeta)^2 + a^2(\alpha^2+b)^2 + \lambda^2} , \qquad (3.1.5)$$

where

$$\zeta = \frac{d}{2\rho} , \quad a^2 = \frac{EI}{\rho} , \quad b = \frac{S}{2EI} , \quad \lambda^2 = \omega^2 - \zeta^2 - a^2 b^2 , \quad \omega^2 = \frac{k}{\rho} .$$

It should be noted that λ is not necessarily positive. It is written in squared form only for convenience. The convolution theorem for Laplace transforms is used to invert equation (3.1.5) with respect to the Laplace-transformation, resulting in

$$W(\alpha,t) = (G+\zeta F)e^{-\zeta t} H(\alpha,t;\lambda) + F e^{-\zeta t} \frac{dH}{dt}$$

$$+ \frac{1}{\rho} \int_0^t Q(\alpha,\tau) e^{-\zeta(t-\tau)} H(\alpha,t-\tau;\lambda) d\tau , \qquad (3.1.6)$$

where we have defined the Fourier transform of the function $h(x,t;\lambda)$ as

$$H(\alpha,t;\lambda) = \frac{\sin[t\sqrt{a^2(\alpha^2+b)^2 + \lambda^2}]}{\sqrt{a^2(\alpha^2+b)^2 + \lambda^2}} , \qquad (3.1.7)$$

with the obvious derivative

$$\frac{dH}{dt} = \cos[t\sqrt{a^2(\alpha^2+b)^2 + \lambda^2}] . \qquad (3.1.8)$$

For the inversion of the Fourier transform the major difficulty is the calculation of the inverse of H, and of $\frac{dH}{dt}$. For clarity these calculations are divided into eight distinct steps:

(1) For the inversion of H, it is necessary to calculate

$$h(x,t;\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\alpha,t;\lambda)e^{i\alpha x}\,d\alpha$$

$$= \frac{t}{\pi}\int_{0}^{\infty} \frac{\sin\sqrt{Z^2+z^2}}{\sqrt{Z^2+z^2}}\cos\alpha x\,d\alpha\ , \qquad (3.1.9)$$

since H is an even function of α. To ease the algebraic manipulations we have defined

$$Z^2 = a^2(\alpha^2 + b)^2t^2 \quad \text{and} \quad z^2 = \lambda^2t^2\ .$$

(2) The inverse of $\frac{dH}{dt}$ is written as

$$\frac{\partial h}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dH}{dt}\,e^{i\alpha x}\,d\alpha$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \cos\sqrt{Z^2 + z^2}\,\cos\alpha x\,d\alpha\ . \qquad (3.1.10)$$

The differentiation under the integral sign with respect to the parameter t is permissible since the integral (3.1.10) converges uniformly for $t\,\varepsilon\,[\gamma,\infty]$, $\gamma > 0$, and hence converges to $\frac{\partial h}{\partial t}$ for $t > 0$ [A2].

(3) A well-known identity [A3] between the circular functions and the cylindrical functions makes it possible to write part of the integrand in (3.1.9) as

$$\frac{\sin\sqrt{Z^2+z^2}}{\sqrt{Z^2+z^2}} = \frac{1}{\sqrt{Z^2+z^2}}\left(\frac{\pi\sqrt{Z^2+z^2}}{2}\right)^{1/2} J_{1/2}(\sqrt{Z^2+z^2})$$

$$= \sqrt{\frac{\pi}{2}}\;\frac{J_{1/2}(\sqrt{Z^2+z^2})}{(Z^2+z^2)^{1/4}} . \qquad (3.1.11)$$

(4) Expression (3.1.11) may be written in terms of Sonine's second finite integral with $\mu = 0$, $\nu = -\frac{1}{2}$ as [A3]:

$$\sqrt{\frac{\pi}{2}}\;\frac{J_{1/2}(\sqrt{Z^2+z^2})}{(Z^2+z^2)^{1/4}} = \sqrt{\frac{\pi}{2}}\; Z^{1/2}\int_0^{\pi/2} J_0(z\sin\theta)\; J_{-1/2}(Z\cos\theta)$$

$$\cdot \sin\theta\cos^{1/2}\theta\, d\theta \qquad (3.1.12)$$

$$= \int_0^{\pi/2} J_0(z\sin\theta)\cos(Z\cos\theta)\sin\theta\, d\theta ,$$

since

$$\sqrt{\frac{\pi}{2}}\sqrt{Z\cos\theta}\, J_{-1/2}\,(Z\cos\theta) = \cos(Z\cos\theta) .$$

(5) In view of these developments the kernel h may now be written as

$$h(x,t;\lambda) = \frac{t}{\pi}\int_0^{\infty}\int_0^{\pi/2} J_0(z\sin\theta)\cos(Z\cos\theta)\sin\theta\cos\alpha x\, d\theta\, d\alpha$$

$$= \frac{t}{\pi}\int_0^{\pi/2} J_0(z\sin\theta)\sin\theta\int_0^{\infty}\cos(Z\cos\theta)\cos\alpha x\, d\alpha\, d\theta , \qquad (3.1.13)$$

since

$$\int_0^\infty \cos(Z\cos\theta)\cos\alpha x\, d\alpha$$

converges uniformly for $\theta \in [0, \gamma]$, $\gamma < \frac{\pi}{2}$ [A2, A4].

(6) The evaluation of the infinite integral [A5] is routine, resulting in

$$\int_0^\infty \cos\left\{[a^2(\alpha^2 + b)^2 t^2]^{1/2}\cos\theta\right\}\cos\alpha x\, d\alpha$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{at\cos\theta}}\cos\left[\frac{\pi}{4} + abt\cos\theta - \frac{x^2}{4at\cos\theta}\right]. \qquad (3.1.14)$$

(7) Equation (3.1.13) now becomes

$$h(x,t;\lambda) = \frac{t}{\pi}\int_0^{\pi/2} J_o(z\sin\theta)(\sin\theta)\,\frac{1}{2}\sqrt{\frac{\pi}{at\cos\theta}}\cos\left(\frac{\pi}{4} + abt\cos\theta - \frac{x^2}{4at\cos\theta}\right)d\theta\ .$$

The final form of h is obtained by making the change of variable

$$u = t\cos\theta\ ,$$

and by the substitution of

$$z = \lambda t\ .$$

The result is

$$h(x,t;\lambda) = \frac{1}{\sqrt{4\pi a}}\int_0^t J_o(\lambda\sqrt{t^2 - u^2})\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au} - abu - \frac{\pi}{4}\right)du. \qquad (3.1.15)$$

(8) An application of Leibitz's rule [A4] in conjunction with the identity

$$\frac{d\,J_o}{dz} = -\,J_1(z)$$

to equation (3.1.15) gives the desired inverse for $\frac{dH}{dt}$, namely

$$\frac{\partial h}{\partial t} = -\frac{\lambda t}{\sqrt{4\pi a}}\int_o^t \frac{1}{(t^2-u^2)^{1/2}}\,J_1(\lambda\sqrt{t^2-u^2})\,\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au}-abu-\frac{\pi}{4}\right)du$$

$$+\frac{1}{\sqrt{4\pi a}}\,\frac{1}{\sqrt{t}}\cos\left(\frac{x^2}{4at}-abt-\frac{\pi}{4}\right) = h_t(x,t;\lambda)\ . \qquad (3.1.16)$$

The expressions for h and h_t along with the convolution theorem may now be used to obtain the inverse Fourier transform of equation (3.1.6). For the interchange of orders of integration in the third term of W it is sufficient to assume that Q is a square integrable function of α for all τ. The solution w finally may be written in the form

$$w(x,t) = e^{-\zeta t}\int_{-\infty}^{\infty}[g(\xi)+\zeta f(\xi)]h(x-\xi,t;\lambda)d\xi$$

$$+\,e^{-\zeta t}\int_{-\infty}^{\infty} f(\xi)h_t(x-\xi,t;\lambda)d\xi$$

$$+\,\frac{1}{\rho}\int_o^t e^{-\zeta(t-\tau)}\int_{-\infty}^{\infty} q(\xi,\tau)h(x-\xi,t-\tau;\lambda)d\xi\,d\tau\ . \qquad (3.1.17)$$

The character of w depends on the functions f, g and q as was indicated earlier. Furthermore, w exhibits a strong dependence on the parameter λ. All these aspects are discussed in the next section.

3.2 Discussion of the Solution

It is convenient to characterize the kernel h and its transform H before embarking on a general discussion of the solution w. The total discourse is divided into four distinct parts.

(1) The restrictions are x and λ arbitrary.

(a) h is a continuous function of t, $t > 0$, since the integrand is continuous in the strip $(0,t] \times (-\infty,\infty)$ of the xt-plane.

(b) h is $O(e^{p_o t})$[A6].

(2) The restrictions are $t > 0$ and λ arbitrary.

(a) h is a continuous function of x, since the integrand is continuous on $(0,t] \times (-\infty,\infty)$ of the xt-plane.

(b) h is square integrable in x since H is absolutely and square integrable. This follows immediately from a special case of Parseval's equality, namely

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha \ .$$

(3) The restrictions are $t > 0$ and x arbitrary. The character of h changes considerably for different values of λ. Hence, it is convenient to define

(i) $S \neq 0,\ k \neq 0,\ d \neq 0$: $\lambda^2 = \lambda_o^2 = \omega^2 - \zeta^2 - a^2b^2$.

(ii) $S = 0,\ k \neq 0,\ d \neq 0$: in general: $\lambda^2 = \lambda_1^2$.

$\omega^2 > \zeta^2$: $\lambda_{1c}^2 = \omega^2 - \zeta^2$

$\omega^2 = \zeta^2$: $\lambda = \lambda_c = 0$.

$\omega^2 < \zeta^2$: $\lambda_{hc}^2 = \zeta^2 - \omega^2$.

(iii) $S = 0,\ k \neq 0,\ d = 0$: $\lambda_2^2 = \omega^2$.

(iv) $S \neq 0,\ k = 0,\ d = 0$: $\lambda_3^2 = -a^2 b^2$.

The variation of h with λ will be discussed only for the case (ii), since this case is analogous to a simple, damped, one-degree of freedom system. The analogy is not complete since the radical to be considered here is $\sqrt{\omega^2 - \zeta^2}$ rather than $\sqrt{\zeta^2 - \omega^2}$. The extension of this discussion to the inclusion of S is obvious. The following definitions are introduced:

(a) Subcritical damping. Here,

$$d < d_c \quad \text{and} \quad \lambda_{1c}^2 = \omega^2 - \zeta^2 \ ,$$

where $d_c = 2\sqrt{k\rho}$, the critical damping coefficient. The expression for h is

$$h(x,t;\lambda_{1c}) = \frac{1}{\sqrt{4\pi a}} \int_0^t J_o(\lambda_{1c}\sqrt{t^2 - u^2}) \frac{1}{\sqrt{u}} \cos\left(\frac{x^2}{4au} - \frac{\pi}{4}\right) du \quad (3.2.1)$$

(b) Critical damping. Now,

$$d = d_c \quad \text{and} \quad \lambda = 0 \ .$$

The function h here has the form it would have if the system contained neither damping nor an elastic foundation, i.e.

$$h(x,t;0) = \frac{1}{\sqrt{4\pi a}} \int_o^t \frac{1}{\sqrt{u}} \cos\left(\frac{x^2}{4au} - \frac{\pi}{4}\right) du \ ,$$

since $J_o(0) = 1$. A change of variable

$$\frac{x}{\sqrt{4au}} = \eta$$

has the result

$$h(x,t;0) = \sqrt{\frac{2}{\pi}}\ \frac{x}{4a}\int_{\frac{x}{\sqrt{4at}}}^{\infty} \frac{\cos\eta^2 + \sin\eta^2}{\eta^2}\, d\eta\ .$$

The integral may be evaluated in the form

$$h(x,t;0) = \sqrt{\frac{t}{\pi a}}\ \cos\left(\frac{x^2}{4at} - \frac{\pi}{4}\right) + \frac{x}{2a}\left[\, \mathfrak{S}\left(\sqrt{\frac{2}{\pi}}\ \frac{x}{\sqrt{4at}}\right) - \mathfrak{C}\left(\sqrt{\frac{2}{\pi}}\ \frac{x}{\sqrt{4at}}\right)\right] \tag{3.2.2}$$

(c) Supercritical damping. In this case

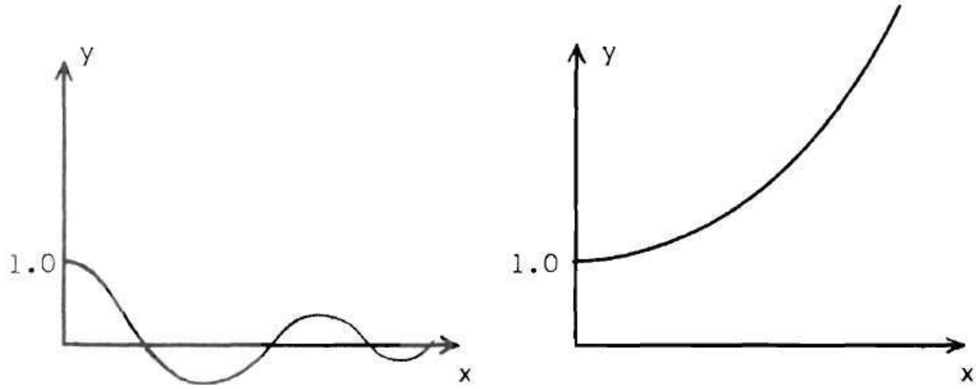
$$d > d_c \quad \text{and} \quad \lambda_{hc}^2 = \zeta^2 - \omega^2\ .$$

The argument of J_o in the integrand becomes purely imaginary, with the result

$$h(x,t;\lambda_{hc}) = \frac{1}{\sqrt{4\pi a}}\int_o^t I_o\left(\lambda_{hc}\sqrt{t^2 - u^2}\right)\frac{1}{\sqrt{u}}\ \cos\left(\frac{x^2}{4au} - \frac{\pi}{4}\right)du, \tag{3.2.3}$$

where I_o is the modified Bessel function of the first kind.

Some indication concerning the behavior of the integrands in equations (3.2.1) and (3.2.3) is given by the schematic sketches of J_o and I_o in Figure 4.



(a) $J_o(x)$ (b) $I_o(x)$

Figure 4. Schematic Sketches of $J_o(x)$ and $I_o(x)$.

(4) It was indicated previously that w could not be characterized completely, unless classifications for the functions f, g, and q had been given. In order to give some idea concerning the character of the solution, sufficient conditions for the existence of the solution in the form (3.1.17) are given here:

(a) (i) Assume that f, g, and q are square integrable in x for every $t \geq \gamma > 0$, and that q is piecewise continuous in t and is $O(e^{p_o t})$ for every x.

(ii) Assume further that the transforms F and G are $o(\alpha^5)$ and that Q is $o(\alpha^3)$.

(b) Consider now the stipulations made in section 3.1.

(i) Due to (a)(i) above, the Laplace transforms with respect to t and the Fourier transforms with respect to x of $w(x,t)$ and $q(x,t)$ exist [A4].

(ii) The Fourier transforms of f and g exist.

(iii) In view of the uniform convergence of the inner integral, and the existence of the transform of w,

$$\mathcal{F}\left\{\frac{\partial w}{\partial t}\right\} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t}\frac{1}{2\pi}\int_{-\infty}^{\infty} W(\alpha,t)e^{i\alpha x}d\alpha\right)e^{-i\alpha x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{dW(\alpha,t)}{dt}\,e^{i\alpha x}\,d\alpha\right)\,e^{-i\alpha x}\,dx$$

$$= \frac{dW(\alpha,t)}{dt}\,.$$

A similar argument applies for the second derivative.

(iv) The limit operation

$$\lim_{t\to 0} w(x,t) = \lim_{t\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} W(\alpha,t)e^{i\alpha x}d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{t\to 0} W(\alpha,t)e^{i\alpha x}d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\,d\alpha = f(x)$$

by uniform convergence. The $\lim\limits_{t\to 0}\dfrac{\partial w(x,t)}{\partial t}$ is treated in exactly the same manner.

(v) In view of (a)(ii)

$\alpha^{\nu}FH$, $\alpha^{\nu}GH$ and $\alpha^{\nu}QH$ are in L for $\nu = 0,1,2,3$.

The Riemann-Lebesgue Lemma [A4] may be applied directly to yield

$$\lim_{|x|\to\infty}\frac{\partial^{\nu} w(x,t)}{\partial x^{\nu}} = 0, \quad \text{for } \nu = 0,1,2,3.$$

CHAPTER IV

COMPARISON OF THE RESULTS

The solution obtained in the previous section is perfectly general within the cited limitations on the loading and on the initial conditions. Hence, it is natural that less general problems solved previously in the literature, are derivable from the present solution by inserting the appropriate loading, initial conditions and material parameters. The primary concern then is to show this inclusiveness rather than presenting detailed discussions of the solutions. These may be found in the cited papers. The delta distribution, $\delta(x)$, is used extensively, since closed form solutions are most accessible for loadings involving $\delta(x)$ and the unit step function $U(x)$. Good discussions of the engineering applications of distributions are found in Stakgold* and in Papoulis.** The comparisons with selected problems of the literature are made in the same order as they appear in the historical introduction.

4.1 The Free Vibration of an Infinite String.

Material parameters: $d = k = 0$; $S \neq 0$; $\lambda = \lambda_3$.

Type of loading: $q = 0$.

Initial conditions: $w_a(x,0) = f(x)$; $\dfrac{\partial w_a(x,0)}{\partial t} = g(x)$.

Order requirements: Same as (2.2.5).

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*Stakgold, T.: Boundary Value Problems of Mathematical Physics. Vol. 1, The MacMillan Company, New York, 1967.

**Papoulis, A.: The Fourier Integral and Its Applications. McGraw-Hill Book Company, Inc., 1962.

The deflection of the string then is given by the limit

$$\lim_{a \to 0} w_a(x,t) = \lim_{a \to 0}\left\{\int_{-\infty}^{\infty} g(\xi)h(x-\xi,t;\lambda_3)d\xi + \int_{-\infty}^{\infty} f(\xi)h_t(x-\xi,t;\lambda_3)d\xi\right\}$$

$$= \lim_{a \to 0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\frac{\sin[t\sqrt{a^2\alpha^4 + c^2\alpha^2}]}{\sqrt{a^2\alpha^4 + c^2\alpha^2}} e^{i\alpha x}d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\cos[t\sqrt{a^2\alpha^4 + c^2\alpha^2}]e^{i\alpha x}d\alpha\right\}. \tag{4.1.1}$$

Since the integrals converge uniformly for every finite a, w_a is a continuous function of a, and

$$w_o(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\frac{\sin(ct\alpha)}{c\alpha}e^{i\alpha x}d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\cos(ct\alpha)e^{i\alpha x}d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2ci\alpha}G(\alpha)[e^{i\alpha(x+ct)} - e^{i\alpha(x-ct)}]d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\frac{1}{2}[e^{i\alpha(x+ct)} + e^{i\alpha(x-ct)}]d\alpha$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi\ . \tag{4.2.2}$$

As usual, we have defined

$$c^2 = \frac{S}{\rho}\ .$$

The solution (4.2.2) conforms precisely with D'Alembert's solution as cited for example by Sneddon.*

4.2 Boussinesq's Solution for the Free Vibration of an Infinite Beam.

Material parameters: $S = k = d = 0$; $\lambda = 0$.

Type of loading: $q = 0$.

Initial conditions: $w(x,0) = f(x)$; $\dfrac{\partial w(x,0)}{\partial t} = a \dfrac{d^2 g_o}{dx^2}$.

Order requirements: Same as (2.2.5).

The general form of the solution is

$$w(x,t) = \int_{-\infty}^{\infty} a \frac{d^2 g_o}{dx^2} h(x-\xi,t;0)d\xi + \int_{-\infty}^{\infty} f(\xi)h_t(x-\xi,t;0)d\xi \ . \qquad (4.2.1)$$

Two integrations by parts in the first integral result in

$$w(x,t) = \int_{-\infty}^{\infty} ag_o(\xi) \frac{\partial^2 h}{\partial \eta^2} d\xi + \int_{-\infty}^{\infty} f(\xi) \frac{\partial h}{\partial t} d\xi \ , \qquad (4.2.2)$$

where $\eta = x - \xi$. The special form of h with $\lambda = 0$ has been given previously as equation (3.2.2). The first derivative of h with respect to x is

$$\frac{\partial h}{\partial x} = \frac{1}{2a} [\mathfrak{S}(\sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{4at}}) - \mathfrak{C}(\sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{4at}})] \ . \qquad (4.2.3)$$

Another differentiation with respect to x yields

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4at}} \frac{1}{a} [\sin \frac{x^2}{4at} - \cos \frac{x^2}{4at}] \ . \qquad (4.2.4)$$

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*Sneddon, I. N.: Fourier Transforms. McGraw-Hill Book Company, Inc., 1951, p. 98.

The substitution of this last expression into equation (4.2.2) has the result

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{4at}}\int_{-\infty}^{\infty} g_o(\xi)\left[\sin\frac{(x-\xi)^2}{4at} - \cos\frac{(x-\xi)^2}{4at}\right]d\xi$$

$$+ \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{4at}}\int_{-\infty}^{\infty} f(\xi)\left[\cos\frac{(x-\xi)^2}{4at} + \sin\frac{(x-\xi)^2}{4at}\right]d\xi. \tag{4.2.5}$$

The change of variable

$$\frac{x-\xi}{\sqrt{4at}} = \alpha$$

yields

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_o(x - 2\alpha\sqrt{at})[\sin\alpha^2 - \cos\alpha^2]d\alpha$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x - 2\alpha\sqrt{at})[\cos\alpha^2 + \sin\alpha^2]d\alpha. \tag{4.2.6}$$

This is the form of the solution cited by Sneddon.* Note that the solution here differs from that of equation (1.16) by a minus sign in the argument of f and g_o. This is remedied by simply replacing α by $-\alpha$.

4.3 The Steady-State Solutions of Ludwig, Dörr and Kenney.

Material parameters: $k \neq 0$, $d \neq 0$, $S = 0$, $\lambda = \lambda_1$.

Type of loading: $q(x,t) = F\delta(x - vt)$.

Initial conditions: $w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0$.

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*Ibid., p. 113.

Order requirements: Same as (2.2.5).

As was indicated previously, we take the steady-state solution to be the result of a formal passage to the limit as t tends to infinity in the transient solution. Usually this limit is taken along the t-axis. However, the "steady-state" solutions of Ludwig, Dörr and Kenney are obtained by taking the limit as t tends to infinity along the lines η = constant in the x, t-plane. This is accomplished by first applying a Gallilean transformation

$$\eta = x - vt, \quad \varepsilon = t$$

to the solution

$$w(x,t) = \frac{F}{\rho}\int_0^t \int_{-\infty}^{\infty} \delta(\xi - v\tau)e^{-\zeta(t-\tau)}h(x-\xi, t-\tau;\lambda_1)d\xi\, d\tau$$

$$= \frac{F}{\rho}\int_0^t e^{-\zeta r}\, h[x - v(t-r), r;\lambda_1]dr\ , \qquad (4.3.1)$$

where we have set $r = t - \tau$. The result of the transformation is

$$w_0(\eta,\varepsilon) = \frac{F}{\rho}\int_0^{\varepsilon} e^{-\zeta r}\, h(\eta + vr, r;\lambda_1)dr\ . \qquad (4.3.2)$$

The evaluation of $\lim_{\varepsilon\to\infty} w_0(\eta,\varepsilon)$ is eased considerably if one first makes a Fourier transformation with respect to η in equation (4.3.2). This ultimately amounts to an interchange of orders of integration of the form

$$\int_{-\infty}^{\infty}\int_0^{\infty} s(\eta,r)dr d\eta = \int_0^{\infty}\int_{-\infty}^{\infty} s(\eta,r)d\eta dr$$

where

$$s(\eta,r) = e^{-\zeta r+i\alpha\eta}\, h(\eta + vr, r; \lambda_1) \ .$$

By [A2] the integral with respect to η converges uniformly for every $r > 0$. The integral with respect to r converges uniformly for all η, since [A6]

$$|s(\eta,r)| < e^{[-\zeta+\sqrt{\zeta^2-\omega^2}]r}\, M \ ,$$

and the procedure is justified. The result is

$$\begin{aligned}
W_o(\alpha,\varepsilon) &= \frac{F}{\rho}\int_0^{\varepsilon} e^{-\zeta r+i\alpha vr}\ \frac{\sin[r\sqrt{a^2\alpha^4+\lambda_1^2}]}{\sqrt{a^2\alpha^4+\lambda_1^2}}\ dr \\
&= \frac{F}{2i\rho\sqrt{a^2\alpha^4+\lambda_1^2}}\left\{\frac{e^{[-\zeta+i(\alpha v+\sqrt{a^2\alpha^4+\lambda_1^2})]\varepsilon}}{-\zeta+i(\alpha v+\sqrt{a^2\alpha^4+\lambda_1^2})}\right. \\
&\qquad \left. -\frac{e^{[-\zeta+i(\alpha v-\sqrt{a^2\alpha^4+\lambda_1^2})]\varepsilon}}{-\zeta+i(\alpha v-\sqrt{a^2\alpha^4+\lambda_1^2})} + \frac{2i\sqrt{a^2\alpha^4+\lambda_1^2}}{a^2\alpha^4-\alpha^2v^2-2\zeta i\alpha v+\omega^2}\right\} . \qquad (4.3.3)
\end{aligned}$$

Recall further that

$$\sqrt{a^2\alpha^4+\lambda_1^2} = \sqrt{a^2\alpha^4+\omega^2-\zeta^2} \ ,$$

so that difficulties can arise only when

$$\zeta^2 > \omega^2 + a^2\alpha^4 \ .$$

For this case, if $\lim_{\varepsilon\to\infty} W_o(\alpha,\varepsilon)$ is to exist, a necessary and sufficient condition is

$$-\zeta + \sqrt{\zeta^2 - a^2\alpha^4 - \omega^2} < 0$$

or

$$-a^2\alpha^4 < \omega^2 ,$$

which is always the case. Hence, it may be concluded that as long as $d > 0$,

$$\lim_{\varepsilon \to \infty} W_o(\alpha, \varepsilon) = W(\alpha) = \frac{F}{\rho} \frac{1}{a^2\alpha^4 - \alpha^2 v^2 - 2\zeta i\alpha v + \omega^2} . \qquad (4.3.4)$$

This is the same expression which would be obtained if the Fourier transform with respect to x were taken of equation (9a) in Kenney's paper. He obtained the inverse transform of equation (4.3.4) in terms of a parameter which was the root of a sixth degree polynomial. He failed to mention however, that in the case of critical damping, i.e. when

$$\left(\frac{d}{2\rho}\right)^2 = \frac{k}{\rho}$$

the denominator in equation (4.3.4) may be factored without resort to the parameter introduced by Kenney. A simple contour integration [A7] and some algebraic manipulation yields the solution in closed form as

$$w_\infty(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_\infty(\alpha) e^{i\alpha\eta} \, d\alpha$$

$$= \frac{F}{8\rho a^2} \frac{1}{\beta^2\sqrt{\beta^2+\gamma^2}} \begin{cases} e^{-\eta\sqrt{\beta^2-\gamma^2}}\left\{ \left(\frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}\right)^{1/2} \cos[\eta(\sqrt{2}\gamma+\sqrt{\beta^2+\gamma^2})] \right. \\ \left. - \frac{\sqrt{2}\gamma\sqrt{\beta^2+\gamma^2}-(\beta^2-\gamma^2)}{\sqrt{2}\gamma\sqrt{\beta^2+\gamma^2}+\sqrt{2}(\beta^2+\gamma^2)} \sin[\eta(\sqrt{2}\gamma+\sqrt{\beta^2+\gamma^2})]\right\} \\ \qquad\qquad \text{for } \eta > 0 , \\ e^{\eta\sqrt{\beta^2-\gamma^2}}\left\{ \left(\frac{\beta^2-\gamma^2}{\beta^2+\gamma^2}\right)^{1/2} \cos[\eta(\sqrt{2}\gamma-\sqrt{\beta^2+\gamma^2})] \right. \\ \left. + \frac{\sqrt{2}\gamma\sqrt{\beta^2+\gamma^2}+(\beta^2-\gamma^2)}{\sqrt{2}\gamma\sqrt{\beta^2+\gamma^2}-\sqrt{2}(\beta^2+\gamma^2)} \sin[\eta(\sqrt{2}\gamma-\sqrt{\beta^2+\gamma^2})]\right\} \\ \qquad\qquad \text{for } \eta < 0 , \end{cases} \tag{4.3.5}$$

where

$$\beta^2 = \frac{1}{2}\sqrt{\left(\frac{v}{2a}\right)^4 + \frac{k}{\rho a^2}} \quad \text{and} \quad \gamma^2 = \frac{1}{2}\left(\frac{v}{2a}\right)^2 .$$

Clearly,

$$|\beta| > |\gamma| \quad \text{and} \quad \sqrt{\beta^2+\gamma^2} > \sqrt{2}\,\gamma ,$$

and it may be concluded that in the case of critical damping the "steady-state" solution has the same character, no matter what the value of v.

The "steady-state" solution obtained by Ludwig and Dörr for $v < v_{cr}$ results when the inverse Fourier transform of equation (4.3.4) is taken with d = 0. One cringes at this thought since it was just shown prior to equation (4.3.4) that d > 0 was a necessary condition for the existence of the "steady-state" solution in the transformed form (4.3.4) as the limit of the transient solution. However, this is an inherent difficulty in vibration problems where damping is not included. Hence, with d = 0, the inverse Fourier transform of (4.3.4) is obtained to be [A7]

$$w_\infty(\eta) = \frac{F}{\rho}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha\eta}\, W_\infty(\alpha)d\alpha$$

$$= \frac{Fe^{-\beta|\eta|}}{4\sqrt{kEI}}\left\{\frac{\sin\gamma|\eta|}{\gamma} + \frac{\cos\gamma|\eta|}{\beta}\right\}, \tag{4.3.6}$$

where

$$\beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{\rho v^2}{4EI}} \quad \text{and} \quad \gamma = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{\rho v^2}{4EI}}\,.$$

As was noted by Ludwig, equation (4.3.6) reduces to the static solution of a beam on an elastic foundation with a concentrated load at the origin when v = 0.* Expression (4.3.6) differs from that obtained by Ludwig in the sign between the sine and cosine terms, and by a β in the denominator of the cosine term. However, since the solution obtained by Dörr is the same as the one obtained here, one is led to conclude that there is a misprint in Ludwig's paper. The steady-state solution for all values of v was discussed exhaustively by Dörr so that any further discussion here would be redundant.

4.4 The Green's Function Obtained by Tseng in Solving the Problem of a Mass Moving on an Infinite Beam.

Material parameters: $S = k = d = 0$; $\lambda = 0$.

Type of loading: $q(x,t) = F(t)$.

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*Kármán, T. and Biot, M. A.: Mathematical Methods in Engineering. McGraw-Hill Book Company, Inc., 1940, p. 272.

Initial conditions: $w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0.$

Order requirements: Same as (2.2.5).

The Green's function obtained by Tseng corresponds to

$$h(\xi - x, \tau - t; 0) = \sqrt{\frac{(t-\tau)}{\pi a}} \cos\left[\frac{(\xi-x)^2}{4a(\tau-t)} - \frac{\pi}{4}\right]$$

$$+ \frac{(\xi - x)}{2a}\left\{\mathfrak{S}\left[\sqrt{\frac{2}{\pi}}\ \frac{(\xi - x)}{\sqrt{4a(\tau-t)}}\right] - \mathfrak{C}\left[\sqrt{\frac{2}{\pi}}\ \frac{(\xi-x)}{\sqrt{4a(\tau-t)}}\right]\right\}. \tag{4.4.1}$$

A regrouping of the variables in the form

$$\eta - y = a(\tau-t) \quad \text{and} \quad \nu = \frac{\xi - x}{\sqrt{\eta - y}} \tag{4.4.2}$$

has the result

$$h_o(\xi-x,\eta-y;0) = \sqrt{\frac{2}{\pi}}\ \frac{\xi - x}{2a\nu}\left[\cos\frac{\nu^2}{4} + \sin\frac{\nu^2}{4}\right]$$

$$+ \frac{(\xi - x)}{2a}\left[\mathfrak{S}\left(\sqrt{\frac{2}{\pi}}\ \frac{\nu}{2}\right) - \mathfrak{C}\left(\sqrt{\frac{2}{\pi}}\ \frac{\nu}{2}\right)\right]. \tag{4.4.3}$$

The definition of the Fresnel integrals used by Tseng is related to the definition used here by the equations

$$\mathfrak{S}(x) = \mathfrak{S}_T\left(\frac{\pi}{2}x^2\right) \quad \text{and} \quad \mathfrak{C}(x) = \mathfrak{C}_T\left(\frac{\pi}{2}x^2\right).$$

With these identities in mind one may finally write

$$h_o(\xi-x,\eta-y,0) = \frac{1}{2\pi a}(\xi - x)\left\{\frac{\sqrt{2\pi}}{v}\left[\cos\frac{v^2}{4} + \sin\frac{v^2}{4}\right] + \pi\left[\mathcal{S}_T\left(\frac{v^2}{4}\right) - \mathcal{C}_T\left(\frac{v^2}{4}\right)\right]\right\}$$

$$= \frac{1}{2\pi a} G(x,y;\xi,\eta) \quad . \tag{4.4.4}$$

The factor $\frac{1}{2\pi a}$ stems from the difference in form of the original equations which were solved.

4.5 Nowacki's Solution for the Response of a Beam on an Elastic Foundation.

Material parameters: $S = d = 0$; $k \neq 0$; $\lambda = \omega$.

Type of loading: $q(x,t) = P_o\delta(x)\delta(t)$.

Initial conditions: $w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0$.

Order requirements: Same as (2.2.5).

In his comprehensive treatment of the application of transform methods to the dynamics of elastic systems, Nowacki obtained the dynamic response to an infinite beam on an elastic foundation subject to the loading stated above. He constructed this solution by taking first the Laplace-transform with respect to t and subsequently the Fourier transform with respect to x. He had no difficulty inverting the Fourier transform; he stated however, that the inversion of the Laplace transform was connected with serious difficulties and finally solves only for x = 0.

The substitution of the conditions stated above into the general solution (3.1.17) results in

$$w(x,t) = \frac{1}{\rho}\int_0^t \int_{-\infty}^{\infty} P_o\delta(\xi)\delta(\tau)h(x-\xi,t-\tau;\omega)d\xi\, d\tau$$

$$= \frac{P_o}{\rho} \int_o^t \delta(\tau) h(x, t-\tau; \omega) d\tau$$

$$= \frac{P_o}{\rho} h(x, t; \omega)$$

$$= \frac{P_o}{\rho \sqrt{4\pi a}} \int_o^t J_o(\omega \sqrt{t^2 - u^2}) \frac{1}{\sqrt{u}} \cos\left(\frac{x^2}{4au} - \frac{\pi}{4}\right) du . \qquad (4.5.1)$$

In the particular case $x = 0$, this expression becomes

$$w(0, t) = \frac{P_o}{\rho \sqrt{4\pi a}} \frac{\sqrt{2}}{2} \int_o^t J_o(\omega \sqrt{t^2 - u^2}) \frac{1}{\sqrt{u}} du . \qquad (4.5.2)$$

The change of variable

$$u = t \cos \theta$$

results in

$$w(0, t) = \frac{P_o}{\rho \sqrt{4\pi a}} \frac{\sqrt{2}}{2} \sqrt{t} \int_o^{\pi/2} J_o(\omega t \sin\theta) \cos^{-1/2}\theta \sin\theta \, d\theta. \qquad (4.5.3)$$

This is Sonine's first integral with $z = \omega t$, $\nu = -\frac{3}{4}$, $\mu = 0$ [A3]. Hence the integral in expression (4.5.3) may be evaluated directly to give

$$w(0, t) = \sqrt{\frac{k}{\rho}} \frac{P_o \Gamma(\frac{1}{4})}{2\sqrt{\pi} (4EI)^{1/4} (2k)^{3/4}} \left(\sqrt{\frac{k}{\rho}} t\right)^{1/4} J_{1/4}\left(\sqrt{\frac{k}{\rho}} t\right). \qquad (4.5.4)$$

This expression differs from that obtained by Nowacki by a factor of $\sqrt{\frac{k}{\rho}}$ and in the replacement of $\Gamma(\frac{3}{4})$ by $\Gamma(\frac{1}{4})$. It is noteworthy that the result (4.5.1) has not previously been contained in the literature.

CHAPTER V

NEW RESULTS, CONCLUSION AND RECOMMENDATIONS

5.1 New Results

The solutions to numerous problems may now be obtained easily by substituting the requisite initial conditions and forcing function into the general solution (3.1.17). A few examples are given here.

(1) A time-dependent concentrated load moving on the beam with an arbitrary velocity.

The initial conditions are assumed to be zero. The loading is

$$q(x,t) = F(t)\delta[x - g(t)] \ .$$

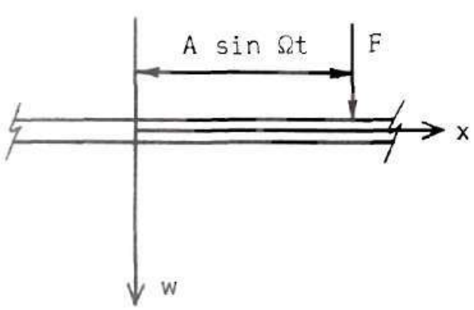
where $g(t)$ may be a continuous function of t, or more generally even, a distribution. The solution is

$$w(x,t) = \frac{1}{\rho}\int_0^t\int_{-\infty}^{\infty} F(\tau)\delta[\xi - g(\tau)]\ e^{-\zeta(t-\tau)}h(x-\xi,t-\tau;\lambda)d\xi d\tau$$

$$= \frac{1}{\rho}\int_0^t F(\tau)e^{-\zeta(t-\tau)}h[x-g(\tau),t-\tau;\lambda]\ d\tau \ . \tag{5.1.1}$$

Some interesting subcases are:

(a)



A constant force F, oscillating with an amplitude A, and frequency Ω about the point $x = 0$. The form of the solution is

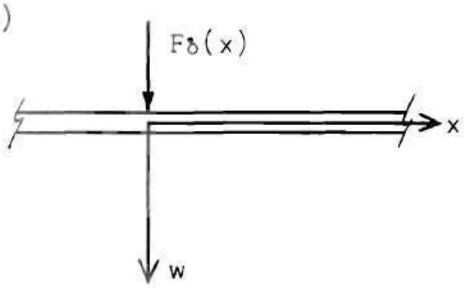
$$w(x,t) = \frac{F}{\rho}\int_0^t e^{-\zeta(t-\tau)}\, h[x - A \sin \Omega\tau, t-\tau;\lambda]\, d\tau \,, \tag{5.1.2}$$

where

$$h[x - A \sin \Omega\ , t-\tau;\lambda] = \frac{1}{\sqrt{4\pi a}}\int_0^{t-\tau} J_o(\lambda\sqrt{(t-\tau)^2 - u^2}\,)$$

$$\cdot \frac{1}{\sqrt{u}} \cos\left[\frac{(x - A \sin \Omega\tau)^2}{4au} - abu - \frac{\pi}{4}\right]du. \tag{5.1.3}$$

(b)



A constant force F applied suddenly at $x = 0$, $t = 0$, and remaining on the beam thereafter. Assume furthermore that $S = 0$. The solution is

$$w(x,t) = \frac{F}{\rho}\int_0^t e^{-\zeta r}\, h(x,r;\lambda_1)dr \,, \tag{5.1.4}$$

where the change of variable

$$t - \tau = r$$

was made. For definiteness consider only $\omega^2 > \zeta^2$. Then,

$$h(x,r;\lambda_1) = \frac{1}{\sqrt{4\pi a}}\int_0^r J_o(\lambda_1\sqrt{r^2 - u^2}\,)\frac{1}{\sqrt{u}} \cos\left(\frac{x^2}{4au} - \frac{\pi}{4}\right) du. \tag{5.1.5}$$

It is instructive here to perform the necessary calculations for the steady-state solution. As indicated previously, the steady-state solution is obtained by passing to the limit as t approaches infinity along the t-axis, i.e.

$$w_\infty(x) = \lim_{t\to\infty} w(x,t)$$

$$= \frac{F}{\rho}\int_0^\infty e^{-\zeta r}\, h(x,r;\lambda_1)dr \;.$$

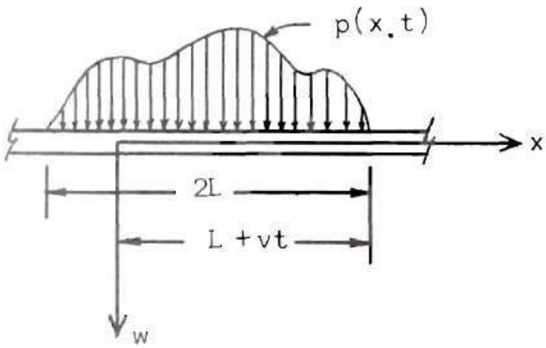
This integral may be interpreted as the Laplace transform of h, and hence may be evaluated to be [A8]

$$w_\infty(x) = \frac{F}{(64\ k^3 EI)^{1/4}}\ e^{-\beta|x|}[\cos\beta|x| + \sin\beta|x|] \;, \qquad (5.1.7)$$

where $\beta = (\frac{k}{4EI})^{1/4}$, and where we have used $\lambda_1{}^2 = \frac{k}{\rho} - \zeta^2$. Equation (5.1.7) is the static deflection of an infinite beam on an elastic foundation, with a concentrated load F at the origin.* This result was to be expected. It is nevertheless gratifying to find one's expectations confirmed.

(2) A time-dependent, moving, distributed load.

Consider an arbitrary loading $p(x,t)$, distributed over a length 2L of the beam, and moving along the beam with a constant velocity v. The loading may be expressed in the form



$$q(x,t) = p(x,t)[U(vt+L) - U(vt-L)] \;,$$

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*Ibid., p. 272.

where U is the unit step function. The resulting deflection is

$$w(x,t) = \frac{1}{\rho}\int_0^t \int_{v\tau-L}^{v\tau+L} p(\xi,\tau)e^{-\zeta(t-\tau)}h(x-\xi,t-\tau;\lambda)d\xi d\tau \,, \qquad (5.1.8)$$

where h has the standard form (3.1.15).

These are just a few of the possible applications of the general solution. The list of possible examples is endless, and complete discussions of solutions are more appropriately given when particular results are desired.

5.2 Conclusion

The problem of the dynamic response of a Bernoulli-Euler beam subjected to an arbitrary loading has been solved, subject to arbitrary initial conditions. Included in the analysis are the effects of an elastic foundation, linear viscous damping and a constant axial force. The form of the solution allows a clear distinction between subcritical, critical, and super-critical damping in terms of the material parameters. It is shown that the general solution includes particular cases obtained previously in the literature, when the requisite choices are made of the constants, the loading, and the initial conditions. In particular it is shown that the steady-state solutions dealt with by several authors are indeed the forms of the solutions obtained by a formal passage to the limit as t tends to infinity in the transient solution. It is shown that a damping coefficient greater than zero is a necessary condition for the existence of the steady-state solution as the limit of the transient solution and that in this manner the solution obtained by Ludwig and Dörr suffers from the usual ailment of problems where damping is neglected, namely that a constant which must be greater than zero for the existence of the solution as a limit has been set equal to zero. Finally, some examples are given of solutions to problems previously unsolved.

5.3 Recommendations

Further investigations in connection with this general solution might be conducted along the following lines:

A. A detailed investigation of the kernel h, including its behavior as a function of the problem parameters.

B. An immediate extension to the problem of an arbitrary force moving on a beam, is the problem of a mass moving on the beam. An attempt at the solution of this problem by transform methods resulted in an expression involving a twelfth degree polynomial in the transform variable. The factorization of this polynomial in closed form was a necessary prerequisite for the inversion of the transformed expression.

C. Resonance and stability investigations for loadings with a periodic character. An example of the occurrence of this phenomenon was the problem solved by Mathews [1958], who found that the beam experienced resonance for certain combinations of v and Ω. We recall that the problem was that of a concentrated load varying in time with frequency Ω, and moving on an infinite beam with an elastic foundation, with constant velocity v.

D. The uniqueness and continuity of the transient and the steady-state solution.

APPENDIX

All statements and algebraic manipulations, which were considered disruptive to the continuity of the main text have been included in the Appendix. This includes comments on the mathematical notation, quotations of theorems in support of claims made in the main body of the thesis, and explanatory remarks.

[A1] Mathematical Notation and Terminology.

(a) The following definitions for the transforms and their inverses will be used throughout

(i) The Fourier transform of a function $f(x,t)$ with respect to x is

$$\mathcal{F}\left\{f(x,t)\right\} = F(\alpha,t) = \int_{-\infty}^{\infty} e^{-i\alpha x} f(x,t)dx \; .$$

(ii) The corresponding inverse transform is

$$\mathcal{F}^{-1}\left\{F(\alpha,t)\right\} = f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x} F(\alpha,t)d\alpha \; .$$

(iii) The Laplace transform of a function $f(x,t)$ with respect to t is

$$\mathcal{L}\left\{f(x,t)\right\} = \bar{f}(x,p) = \int_{0}^{\infty} e^{-pt} f(x,t)dt \; .$$

(iv) For all cases considered in this thesis the corresponding inverse transform may be obtained from

$$\mathcal{L}^{-1}\left\{\bar{f}(x,p)\right\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\bar{f}(x,p)dp \ .$$

(b) A function f is said to be $L^p(a,b)$ if f is measurable and $\int_a^b |f(x)|^p\,dx < \infty$. In this thesis the interval (a,b) will be $(-\infty,\infty)$ unless specified otherwise. Furthermore, L will be used instead of L^1. Since absolutely integrable (L) and square integrable (L^2) are also in use, we shall use these definitions interchangeably here.

(c) The function $f(x,t)$ will be said to be of class C^n in x if f is continuous in x and has continuous derivatives with respect to x up to and including the n^{th}.

(d) As is usual,

(i) $u(x) = O[v(x)]$, u is of order v implies

$$\lim_{x\to\infty} \frac{u(x)}{v(x)} < \infty \ .$$

(ii) $u(x) = o[v(x)]$ implies

$$\lim_{x\to\infty} \frac{u(x)}{v(x)} = 0 \ .$$

(e) The notations,

$Re[z]$ - real part of z,

$Im[z]$ - imaginary part of z,

will be used.

(f) The Fresnel integrals can be defined in a number of different ways.

We shall use the definitions*

$$\mathfrak{C}(x) = \int_0^x \cos\frac{\pi}{2} t^2\, dt \quad \text{and} \quad \mathfrak{S}(x) = \int_0^x \sin\frac{\pi}{2} t^2 dt\ ,$$

so that

$$\mathfrak{S}(\infty) = \mathfrak{C}(\infty) = \frac{1}{2} = -\mathfrak{S}(-\infty) = -\mathfrak{C}(-\infty).$$

[A2] Theorem on Uniform Convergence.

The following theorem is of considerable use in testing the uniform convergence of trigonometric integrals. In form, and in the method of proof it is similar to Dirichlet's test for uniform convergence.

Theorem: Let $f(\alpha)$ be a function, whose first derivative is a continuous, unbounded, monotonically increasing function of α for $\alpha \geq A_1 > 0$. Then, the integral

$$I(x,t) = \int_0^\infty \cos[tf(\alpha)]\ \cos(x\alpha)d\alpha$$

converges uniformly for $t \in [\gamma, \infty)$, $\gamma > 0$, for every fixed x.

Proof: (1) Preliminary considerations

Let $\gamma > 0$ be given and let x be fixed. Choose $A_2 > 0$ such that $\gamma f'(\alpha) > |x|$ for $\alpha \geq A_2$. Take $\alpha_0 = \max\,[A_1, A_2]$, and note that

$$\int_{\alpha_0}^\infty \cos[tf(\alpha)]\cos(x\alpha)d\alpha = \frac{1}{2}\int_{\alpha_0}^\infty \cos[tf(\alpha) + x\alpha]d\alpha$$

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*Gröbner, W., Hofreiter, N.: Integraltafel. Part I: Unbestimmte Integrale; Part II: Bestimmte Integrale, Springer Verlag, Wien, 1961, p. 135 of Part I.

$$+ \frac{1}{2}\int_{\alpha_o}^{\infty} \cos[tf(\alpha) - x\alpha]d\alpha \; .$$

It suffices to consider then

$$\int_{\alpha_o}^{\infty} \cos[tf(\alpha) \pm \alpha x]d\alpha = \int_{\alpha_o}^{\infty} g(\alpha,t)d[h(\alpha,t)]$$

where we have defined

$$g(\alpha,t) = \frac{1}{tf'(\alpha) \pm x} \; , \; h(\alpha,t) = \sin[tf(\alpha) \pm \alpha x]$$

for ease in the following discussion.

In view of these definitions of g and h it follows that

$$|g(\alpha_1,t)| \leq |g(\alpha,t)| \quad \text{for} \quad \alpha_1 \geq \alpha \geq \alpha_o \; ,$$

$$|g(\alpha,t)| \leq |g(\alpha,\gamma)| \quad \text{for} \quad t \geq \gamma > 0 \; ,$$

$$|h(\alpha,t)| \leq 1, \quad \text{for all } \alpha \text{ and } t \; .$$

(2) Proof proper.

Let $\varepsilon > 0$ be given. Let $A_3 > 0$ be such that

$$|g(u,\gamma)| \leq \frac{\varepsilon}{4} \text{ for } u \geq A_3 \; ,$$

and choose $A_\varepsilon = \max[\alpha_o, A_3]$ and $v > u \geq A_\varepsilon$. Then

$$\int_u^v g\,dh = g(v,t)h(v,t) - g(u,t)h(u,t) + \int_u^v h\,d(-g) \; .$$

Since $(-g)$ is an increasing function of α for each t

$$\left|\int_u^v h\,d(-g)\right| \leq \int_u^v d(-g) = g(u,t) - g(v,t)\,.$$

Hence, we finally have

$$\begin{aligned}\left|\int_u^v g\,dh\right| &\leq g(v,t) + g(u,t) + g(u,t) + g(v,t)\\ &\leq 2g(u,\gamma) + 2g(v,\gamma)\\ &\leq 4g(u,\gamma) \leq \varepsilon\,,\end{aligned}$$

and I converges uniformly on $t \geq \gamma > 0$ by the uniform Cauchy condition.

[A3] Some Useful Identities Involving Bessel Functions.

(1)* Bessel functions and circular functions

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z\,,$$

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z\,.$$

(2)** Sonine's first finite integral in general form

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^{\nu}\Gamma(\nu+1)}\int_0^{\pi/2} J_{\mu}(z\sin\theta)\sin^{\mu+1}\theta\,\cos^{2\nu+1}\theta\,d\theta\,.$$

The equality is valid for all z, subject to the conditions Re[μ] and Re[ν] greater than -1.

(3)*** Sonine's second finite integral in general form.

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*Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second Edition, Cambridge at the University Press, 1944, p. 54.

**Ibid., p. 373.

***Ibid., p. 376.

$$\int_0^{\pi/2} J_\mu(z \sin\theta)\, J_\nu(Z\cos\theta) \sin^{\mu+1}\theta \cos^{\nu+1}\theta\, d\theta$$

$$= \frac{z^\mu Z^\nu J_{\mu+\nu+1}\left\{\sqrt{Z^2 + z^2}\right\}}{(Z^2 + z^2)^{1/2(\mu+\nu+1)}} \quad .$$

The equality is valid for all Z and z, as long as Re[μ] and Re[ν] exceed -1.

[A4] Some Useful Theorems.

The theorems quoted here give sufficiency conditions in support of statements made in the main text.

(1)* Theorem: Assume that α is of bounded variation on [a,b] for every b ≥ a and let β be of bounded variation on [c,d]. If f is continuous on the rectangular strip [α, +∞] x [c,d], then f is continuous on [c,d] and we have

$$\int_c^d [\int_a^\infty f(x,y)d\alpha(x)]d\beta(y) = \int_a^\infty [\int_c^d f(x,y)d\beta(y)]d\alpha(x) \ .$$

(2)** Theorem: (Leibnitz's rule). Let $R = \left\{(x,y) | a \le x \le b,\ c \le y \le d\right\}$. Assume that f and $D_2 f$ are continuous on R. Let p and q be two functions having finite derivatives p' and q' on [c,d] and assume that $a \le p(y) \le b$ and that $a \le q(y) \le b$ for each y in [c,d]. Define F by the equation

$$F(y) = \int_{p(y)}^{q(y)} f(x,y)dx, \quad \text{if } y \in [c,d] \ .$$

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*Apostol, T. M.: Mathematical Analysis. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1960, p. 445.

**Ibid., p. 220.

Then $F'(y)$ exists for each y in (c,d) and is given by

$$F'(y) = \int_{p(y)}^{q(y)} D_2 f(x,y)dx + f[q(y),y]q'(y) - f[p(y),y]p'(y) .$$

(3)* Theorem: If f and g belong to $L^2(-\infty,\infty)$, then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)f(x-u)du, \quad FG$$

are transforms in the sense that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)G(t)e^{-ixt}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-ixt}dt \int_{-\infty}^{\infty} g(u)e^{itu}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)du \int_{-\infty}^{\infty} F(t)e^{-it(x-u)}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)f(x-u)du$$

holds for all x.

(4)** Theorem: (Riemann-Lebesgue Lemma). Let f belong to $L(-\infty,\infty)$. Then the integrals

$$\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx, \int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx$$

tend to zero as $\lambda \to \infty$.

[A5] Evaluation of an Integral.

It is easily verified that***

$$\int \cos(ax^2+2bx+c)dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{ac-b^2}{a}\right)\mathfrak{C}\left(\frac{\sqrt{2}(ax+b)}{\sqrt{a\pi}}\right)\right.$$
$$\left. - \sin\left(\frac{ac-b^2}{a}\right)\mathfrak{S}\left(\frac{\sqrt{2}(ax+b)}{\sqrt{a\pi}}\right)\right\} + C.$$

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*Titchmarsh, E. C.: Introduction to the Theory of Fourier Integrals. Second Edition, Oxford at the Clarendon Press, 1948, p. 90.

**Ibid., p. 11

***Groebner and Hofreiter, Op. cit., p. 136 of Part I.

Hence,

$$\int_0^\infty \cos(ax^2 + 2bx + c)dx = \frac{1}{2}\int_{-\infty}^\infty \cos(ax^2 + 2bx + c)dx$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{ac - b^2}{a}\right)\left(\frac{1}{2} + \frac{1}{2}\right) - \sin\left(\frac{ac - b^2}{a}\right)\left(\frac{1}{2} + \frac{1}{2}\right)\right\}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\ \cos\left(\frac{\pi}{4} + \frac{ac - b^2}{a}\right).$$

Equation (3.1.14) is then obtained by the substitution of the requisite variables.

[A6] Order-properties of h.

We consider two separate cases, namely $\lambda = \lambda_{1c}$ and $\lambda = \lambda_{hc}$.

(1) In the first case $\lambda = \lambda_{1c}$. It suffices to consider $t \geq 1$.

$$\left|e^{-p_o t}\int_1^t J_o(\lambda_{1c}\sqrt{t^2 - u^2})\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au} - abu - \frac{\pi}{4}\right)du\right|$$

$$\leq e^{-p_o t}\ (t - 1)\ ,$$

which tends to zero as t tends to infinity for every $p_o > 0$.

(2) In this case $\lambda = \lambda_{hc}$.

$$\left|e^{-p_o t}\int_0^t I_o(\lambda_{hc}\sqrt{t^2 - u^2})\frac{1}{\sqrt{u}}\cos\left(\frac{x^2}{4au} - abu - \frac{\pi}{4}\right)du\right|$$

$$\leq e^{-p_o t}\int_0^t I_o(\lambda_{hc}\ \sqrt{t^2 - u^2})\frac{1}{\sqrt{u}}\ du$$

$$= e^{-p_o t}\sqrt{t}\int_0^{\pi/2} I_o(\lambda_{hc} t \sin\theta)\cos^{1/2}\theta\sin\theta\ d\theta$$

$$= e^{-p_o t} \left(\frac{t}{\lambda_{hc}}\right)^{1/4} I_{1/4}(\lambda_{hc} t) \ .$$

An asymptotic expansion of I_v for large t is*

$$I_v(z) = e^z(2\pi z)^{-1/2}\left[\sum_{k=o}^{n} (-1)^k(v,k)(2z)^{-k} + O(|z|^{-n-1})\right],$$

$$|\arg z| \leq \frac{\pi}{2} - \delta \ .$$

If $z = \lambda_{hc}t$ and $v = \frac{1}{4}$ in this expression, we may again conclude that h is of exponential order in t for $p_o > \lambda_{hc}$.

[A7] Algebraic Details in the Construction of the "Steady-state" Solutions of Ludwig, Dörr and Kenney.

(1) Kenney's omission.

From equation (4.3.4) of the text

$$W_\infty(\alpha) = \frac{F}{a^2\rho}\,\frac{1}{\alpha^4 - \frac{v^2}{a^2}\alpha^2 - 2i\alpha\,\frac{v}{a}\sqrt{\frac{k}{a^2\rho}} + \frac{k}{a^2\rho}}$$

$$= \frac{F}{a^2\rho}\, f(\alpha) \ .$$

Residue calculus is used to evaluate

$$\int_{-\infty}^{\infty} e^{i\eta\alpha}\, W_\infty(\alpha)d\alpha \ .$$

In line therewith we shall integrate $f(z)e^{i\eta z}$ along the contour C_1

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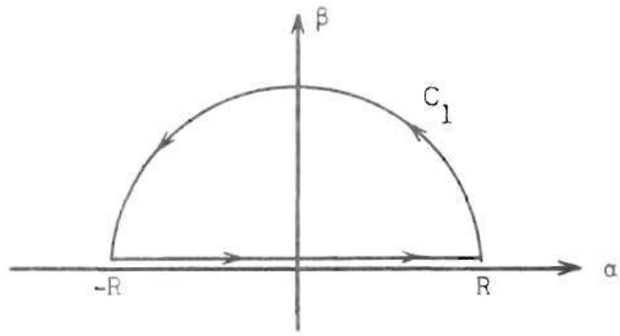
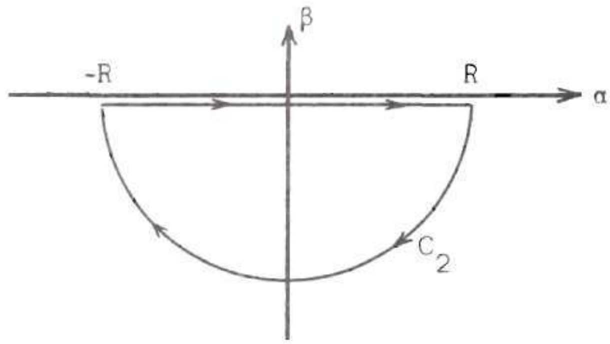
*Lebedev, N. N.: <u>Special Functions and Their Applications</u>. Prentice-Hall, Inc., 1965, p. 123.



for $\eta > 0$, and along C_2



for $\eta < 0$. These restrictions on η insure that $\lim_{R \to \infty}$ of the integrals along the curved parts are zero. Naturally $z = \alpha + \beta i$. The polynomial

$$P(z) = z^4 - \frac{v^2}{a^2} z^2 - 2iz \frac{v}{a}\sqrt{\frac{k}{a^2 \rho}} + \frac{k}{a^2 \rho}$$

may be written in factored form as

$$\begin{aligned} P(z) &= \left[(z - \frac{v}{2a})^2 - (\frac{v^2}{4a^2} + i\sqrt{\frac{k}{\rho a^2}})\right]\left[(z + \frac{v}{2a})^2 - (\frac{v^2}{4a^2} - i\sqrt{\frac{k}{\rho a^2}})\right] \\ &= (z - r_1)(z - r_2)(z - r_3)(z - r_4) \ , \end{aligned}$$

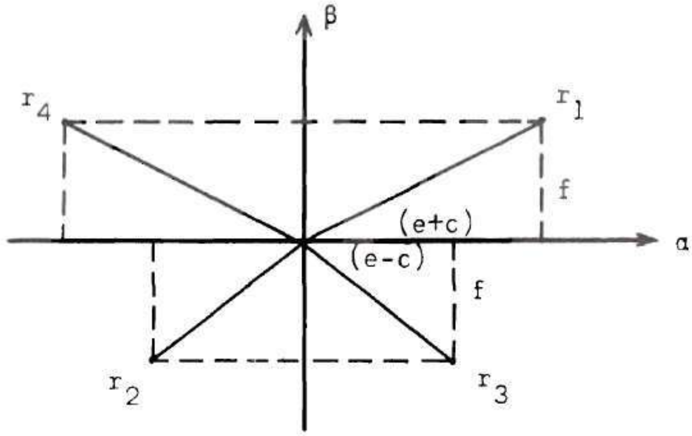
where

$r_1 = (e + c) + if,$

$r_2 = -(e - c) - if,$

$r_3 = (e - c) - if,$

$r_4 = -(e + c) + if,$



with

$$e = [\tfrac{1}{2}(c^4 + g^2)^{1/2} + \tfrac{1}{2} c^2]^{1/2}, \quad f = [\tfrac{1}{2}(c^4 + g^2)^{1/2} - \tfrac{1}{2} c^2]^{1/2}$$

and

$$c = \frac{v}{2a}, \quad g = \sqrt{\frac{k}{\rho a^2}}.$$

The particular form of the sum of e and c was chosen, since for $k > 0$

$$e > c, \text{ i.e. } \frac{v}{2a} < \frac{1}{\sqrt{2}} \sqrt{\sqrt{(\frac{v}{2a})^4 + \frac{k}{\rho a^2}} + (\frac{v}{2a})^2} \quad .$$

The steady state solution then is given by

$$w_\infty(\eta) = \frac{F}{\rho a^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\eta\alpha}\, d\alpha}{P(\alpha)}$$

$$= \frac{F}{\rho a^2}\, i \begin{cases} \sum \text{Res. of } \dfrac{e^{i\eta z}}{P(z)} \text{ of the poles interior to } C_1 \text{ for } \eta > 0, \\ -\sum \text{Res. of } \dfrac{e^{i\eta z}}{P(z)} \text{ of the poles interior to } C_2 \text{ for } \eta < 0. \end{cases}$$

In terms of the r_i these residues are

$$\sum \text{Res.}\ \frac{e^{i\eta z}}{P(z)} = \frac{e^{i\eta r_1}}{(r_1-r_2)(r_1-r_3)(r_1-r_4)} + \frac{e^{i\eta r_4}}{(r_4-r_1)(r_4-r_2)(r_4-r_3)}$$

for $\eta > 0$, and

$$\sum \text{Res.}\ \frac{e^{i\eta z}}{P(z)} = \frac{e^{i\eta r_2}}{(r_2-r_1)(r_2-r_3)(r_2-r_4)} + \frac{e^{i\eta r_3}}{(r_3-r_1)(r_3-r_2)(r_3-r_4)}$$

for $\eta < 0$. A considerable amount of algebraic manipulation yields the solution in its final form as

$$w(\eta) = \frac{F}{8\rho a^2}\,\frac{1}{\beta^2\sqrt{\beta^2+\gamma^2}}\begin{cases} e^{-\eta\sqrt{\beta^2-\gamma^2}}\left\{\left(\dfrac{\beta^2-\gamma^2}{\beta^2+\gamma^2}\right)^{1/2}\cos[\eta(\sqrt{2}\,\gamma+\sqrt{\beta^2+\gamma^2})]\right. \\[2ex] \left. -\dfrac{\sqrt{2}\,\gamma\sqrt{\beta^2+\gamma^2}-(\beta^2-\gamma^2)}{\sqrt{2}\,\gamma\sqrt{\beta^2+\gamma^2}+\sqrt{2}(\beta^2+\gamma^2)}\sin[\eta(\sqrt{2}\,\gamma+\sqrt{\beta^2+\gamma^2})]\right\} \\[2ex] \qquad\qquad\qquad\qquad \text{for } \eta > 0\,, \\[2ex] e^{\eta\sqrt{\beta^2-\gamma^2}}\left\{\left(\dfrac{\beta^2-\gamma^2}{\beta^2+\gamma^2}\right)^{1/2}\cos[\eta(\sqrt{2}\,\gamma-\sqrt{\beta^2+\gamma^2})]\right. \\[2ex] \left. +\dfrac{\sqrt{2}\,\gamma\sqrt{\beta^2+\gamma^2}+(\beta^2-\gamma^2)}{\sqrt{2}\,\gamma\sqrt{\beta^2+\gamma^2}-\sqrt{2}(\beta^2+\gamma^2)}\sin[\eta(\sqrt{2}\,\gamma-\sqrt{\beta^2+\gamma^2})]\right\} \\[2ex] \qquad\qquad\qquad\qquad \text{for } \eta < 0\,, \end{cases}$$

where

$$\beta^2 = \frac{1}{2}\sqrt{\left(\frac{v}{2a}\right)^4 + \frac{k}{\rho a^2}}\,,\quad \gamma^2 = \frac{1}{2}\left(\frac{v}{2a}\right)^2\,,$$

and where it should be kept in mind that

$$(\frac{d}{2\rho})^2 = \frac{k}{\rho} .$$

(2) Ludwig's solution by Fourier transforms.

It will be shown here that the solution which Ludwig obtained by a passage to the limit as the beam length tends to infinity may be obtained in a straight forward manner by the Fourier transform technique.

The expression for the transform of the steady state solution is

$$W_\infty(\alpha) = \frac{F}{EI} \frac{1}{\alpha^4 - \frac{\rho v^2}{EI}\alpha^2 + \frac{k}{EI}} .$$

The denominator may be judiciously factored and written as

$$\alpha^4 - \frac{\rho v^2}{EI}\alpha^2 + \frac{k}{EI} = [\alpha^2 + (\beta + i\gamma)^2][\alpha^2 + (\beta - i\gamma)^2] ,$$

where

$$\beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{\rho v^2}{4EI}} \quad \text{and} \quad \gamma = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{\rho v^2}{4EI}} .$$

A partial fraction expansion is now used to write

$$W_\infty(\alpha) = \frac{F}{EI}\frac{1}{\beta\gamma}\left\{-\frac{1}{4i}\frac{1}{\alpha^2 + (\beta + i\gamma)^2} + \frac{1}{4i}\frac{1}{\alpha^2 + (\beta - i\gamma)^2}\right\}.$$

The inverse Fourier transform is*

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*Bateman, H.: Tables of Integral Transforms. Vol. 1, McGraw-Hill Book Company, Inc., 1954, p. 118.

$$w_\infty(\eta) = \frac{F}{EI}\frac{1}{\beta\gamma}\left\{-\frac{1}{4i}\frac{1}{2(\beta+i\gamma)}e^{-(\beta+i\gamma)|\eta|} + \frac{1}{4i}\frac{1}{2(\beta-i\gamma)}e^{-(\beta-i\gamma)|\eta|}\right\}$$

$$= \frac{F}{EI}\frac{e^{-\beta|\eta|}}{4\beta\gamma(\beta^2+\gamma^2)}\left\{\beta\sin\gamma|\eta| + \gamma\cos\gamma|\eta|\right\}$$

$$= \frac{F\,e^{-\beta|\eta|}}{4\sqrt{kEI}}\left\{\frac{\sin\gamma|\eta|}{\gamma} + \frac{\cos\gamma|\eta|}{\beta}\right\},$$

since

$$\beta^2 + \gamma^2 = \sqrt{\frac{k}{EI}}\ .$$

[A8] Evaluation of the Integral in Equation (5.1.6).

One of the general formulae in the theory of Laplace transforms is*

$$\mathcal{L}\left\{\int_0^t J_0(a\sqrt{t^2-u^2})\,f(u)du\right\} = \frac{1}{(p^2+a^2)^{1/2}}\,\bar{f}[(p^2+a^2)^{1/2}]\ .$$

Furthermore, the Laplace transform of

$$f(u) = \frac{1}{\sqrt{u}}\left[\cos\frac{\lambda}{u} + \sin\frac{\lambda}{u}\right]$$

is**

$$\bar{f}(p) = \sqrt{\frac{\pi}{p}}\,e^{-\sqrt{2p\lambda}}\left[\cos\sqrt{2p\lambda} + \sin\sqrt{2p\lambda}\right].$$

These two formulae are used in conjunction to calculate the transform of

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*Ibid., p. 227.

**Doetsch, G.: Enführung in Theorie und Anwendung der Laplace-Transformation, Birkhäuser Verlag Basel und Stuttgart, 1958, p. 33.

$$\int_0^t J_0(\lambda\sqrt{t^2 - u^2})\frac{1}{\sqrt{u}}\left\{\cos\frac{x^2}{4au} + \sin\frac{x^2}{4au}\right\} du$$

as

$$\frac{1}{(p^2+\lambda^2)^{1/2}}\sqrt{\frac{\pi}{(p^2+\lambda^2)^{1/2}}}\, e^{-\sqrt{2(\frac{x^2}{4a})(p^2+\lambda^2)^{1/2}}}\left\{\cos\sqrt{2(\frac{x^2}{4a})(p^2+\lambda^2)^{1/2}}\right.$$

$$\left. + \sin\sqrt{2(\frac{x^2}{4a})(p^2+\lambda^2)^{1/2}}\right\}.$$

The desired result then is obtained by evaluating at $p = \zeta$.

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VITA

Wolfram Stadler was born in Straubing on the Danube, Niederbaiern, Germany on July 4, 1937. During World War II he attended elementary school in Traun, near Linz, in Austria. At the end of the war, after some short stays in Munich and Regensburg, he finally continued his elementary school education at the "Evangelische Volksschule" in Weiden, Oberpfalz. In 1947 he passed the examination to enter the "Oberrealschule Weiden" where he studied until 1950. He finally finished his "Mittlere Reife" at the "Fuerstenberger Mittelschule" in Frankfurt on the Main in 1954. After working for one year as a supervisor in a youth center and as a proofreader for "The Overseas Weekly" he emigrated to the United States in May 1955. After a short stay with his sponsors in Columbia, Tennessee, he enlisted in the United States Air Force, from which he was honorably discharged in 1959.

From January 1956 to December 1957 he attended night school at the University of Tennessee in Knoxville, Tennessee. In September 1959 he became a full-time student in the School of Aerospace Engineering at Georgia Tech, from which he received his B.S. in Aerospace Engineering in 1963. In 1964 he received his M.S. in Aerospace Engineering. He then enrolled in the School of Engineering Mechanics as a doctoral student. He received his M.S. in Engineering Mechanics in 1966.

In the Air Force Wolfram worked as a draftsman and commercial artist. During all of his time in school he translated technical articles from the German and in the years 1962 to 1965 he taught as a teaching assistant in the School of Applied Mathematics. From 1962-1963 he also taught German at The Lovett School, a private high-school in Atlanta, Georgia. In the period 1965-1967 he was a graduate research assistant and teaching assistant in the School of Engineering Mechanics, and an instructor in the same school from 1966 to 1968.